

A tighter Z -eigenvalue localization set for tensors and its applications

Jianxing Zhao*

*College of Data Science and Information Engineering, Guizhou Minzu University,
Guiyang 550025, P.R. China*

Abstract. A new Z -eigenvalue localization set for tensors is given and proved to be tighter than those presented by Wang *et al.* (Discrete and Continuous Dynamical Systems Series B 22(1): 187-198, 2017) and Zhao (J. Inequal. Appl., to appear, 2017). As an application, a sharper upper bound for the Z -spectral radius of weakly symmetric nonnegative tensors is obtained. Finally, numerical examples are given to verify the theoretical results.

Keywords: Z -eigenvalue; localization set; nonnegative tensors; spectral radius; weakly symmetric

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1 Introduction

For a positive integer n , $n \geq 2$, N denotes the set $\{1, 2, \dots, n\}$. $\mathbb{C}(\mathbb{R})$ denotes the set of all complex (real) numbers. We call $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ a real tensor of order m dimension n , denoted by $\mathbb{R}^{[m, n]}$, if

$$a_{i_1 i_2 \dots i_m} \in \mathbb{R},$$

where $i_j \in N$ for $j = 1, 2, \dots, m$. \mathcal{A} is called nonnegative if $a_{i_1 i_2 \dots i_m} \geq 0$. $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m, n]}$ is called symmetric [1] if

$$a_{i_1 \dots i_m} = a_{\pi(i_1 \dots i_m)}, \quad \forall \pi \in \Pi_m,$$

where Π_m is the permutation group of m indices. $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}^{[m, n]}$ is called weakly symmetric [2] if the associated homogeneous polynomial

$$\mathcal{A}x^m = \sum_{i_1, i_2, \dots, i_m \in N} a_{i_1 i_2 \dots i_m} x_{i_1} x_{i_2} \dots x_{i_m}$$

satisfies $\nabla \mathcal{A}x^m = m\mathcal{A}x^{m-1}$. It is shown in [2] that a symmetric tensor is necessarily weakly symmetric, but the converse is not true in general.

Given a tensor $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m, n]}$, if there are $\lambda \in \mathbb{C}$ and $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{C}^n \setminus \{0\}$ such that

$$\mathcal{A}x^{m-1} = \lambda x \text{ and } x^T x = 1,$$

*Corresponding author. E-mail: zjx810204@163.com; zhaojianxing@gzmu.edu.cn (Jianxing Zhao)

then λ is called an E -eigenvalue of \mathcal{A} and x an E -eigenvector of \mathcal{A} associated with λ , where $\mathcal{A}x^{m-1}$ is an n dimension vector whose i th component is

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, \dots, i_m \in N} a_{ii_2 \dots i_m} x_{i_2} \cdots x_{i_m}.$$

If λ and x are all real, then λ is called a Z -eigenvalue of \mathcal{A} and x a Z -eigenvector of \mathcal{A} associated with λ ; for details, see [1, 3]. Here, we define the Z -spectrum of \mathcal{A} , denoted $\sigma(\mathcal{A})$ to be the set of all Z -eigenvalues of \mathcal{A} . Assume $\sigma(\mathcal{A}) \neq 0$, then the Z -spectral radius [2] of \mathcal{A} , denoted $\varrho(\mathcal{A})$, is defined as

$$\varrho(\mathcal{A}) := \sup\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}.$$

Recently, many people have focused on locating all Z -eigenvalues of tensors and bounding the Z -spectral radius of nonnegative tensors in [2, 4–11]. In 2017, Wang *et al.* [4] established the following Geršgorin-type Z -eigenvalue inclusion theorem for tensors.

Theorem 1. [4, Theorem 3.1] *Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m, n]}$. Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) = \bigcup_{i \in N} \mathcal{K}_i(\mathcal{A}),$$

where

$$\mathcal{K}_i(\mathcal{A}) = \{z \in \mathbb{C} : |z| \leq R_i(\mathcal{A})\}, \quad R_i(\mathcal{A}) = \sum_{i_2, \dots, i_m \in N} |a_{ii_2 \dots i_m}|.$$

To get a tighter Z -eigenvalue inclusion set than $\mathcal{K}(\mathcal{A})$, Wang *et al.* [4] gave the following Brauer-type Z -eigenvalue localization set for tensors.

Theorem 2. [4, Theorem 3.2] *Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m, n]}$. Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}) = \bigcup_{i \in N} \bigcap_{j \in N, j \neq i} \mathcal{L}_{i,j}(\mathcal{A}),$$

where

$$\mathcal{L}_{i,j}(\mathcal{A}) = \{z \in \mathbb{C} : (|z| - (R_i(\mathcal{A}) - |a_{ij \dots j}|))|z| \leq |a_{ij \dots j}|R_j(\mathcal{A})\}.$$

Very recently, Zhao [5] presented another Brauer-type Z -eigenvalue localization set for tensors and proved that this set is tighter than those in Theorem 1 and Theorem 2.

Theorem 3. [5, Theorem 3] *Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m, n]}$. Then*

$$\sigma(\mathcal{A}) \subseteq \Psi(\mathcal{A}) = \bigcup_{i \in N} \bigcap_{j \in N, j \neq i} \Psi_{i,j}(\mathcal{A}),$$

where

$$\Psi_{i,j}(\mathcal{A}) = \left\{ z \in \mathbb{C} : (|z| - r_i^{\bar{\Delta}^j}(\mathcal{A}))|z| \leq r_i^{\Delta^j}(\mathcal{A})R_j(\mathcal{A}) \right\},$$

$$r_i^{\Delta^j}(\mathcal{A}) = \sum_{j \in \{i_2, \dots, i_m\}} |a_{ii_2 \dots i_m}|, \quad r_i^{\bar{\Delta}^j}(\mathcal{A}) = \sum_{j \notin \{i_2, \dots, i_m\}} |a_{ii_2 \dots i_m}|.$$

As we know, one can use eigenvalue inclusion sets to obtain the upper bound of the spectral radius of nonnegative tensors; for details, see [4, 12–15]. Therefore, the main aim of this paper is to give a new Z -eigenvalue inclusion set for tensors and prove that the new set is tighter than those in Theorems 1-3. And as an application, a new upper bound for the Z -spectral radius of weakly symmetric nonnegative tensors is obtained and proved to be sharper than some existing upper bounds.

2 Main results

In this section, we give a new Brauer-type Z -eigenvalue localization set for tensors, and establish the comparison between the new set with those in Theorems 1-3.

Theorem 4. Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m, n]}$. Then

$$\sigma(\mathcal{A}) \subseteq \Omega(\mathcal{A}) = \left(\bigcup_{i \in N} \bigcap_{j \in N, j \neq i} \hat{\Omega}_{i,j}(\mathcal{A}) \right) \cup \left(\bigcup_{i \in N} \bigcap_{j \in N, j \neq i} (\tilde{\Omega}_{i,j}(\mathcal{A}) \cap \mathcal{K}_i(\mathcal{A})) \right),$$

where

$$\hat{\Omega}_{i,j}(\mathcal{A}) = \left\{ z \in \mathbb{C} : |z| < r_i^{\bar{\Delta}^j}(\mathcal{A}), |z| < r_j^{\Delta^j}(\mathcal{A}) \right\}$$

and

$$\tilde{\Omega}_{i,j}(\mathcal{A}) = \left\{ z \in \mathbb{C} : (|z| - r_i^{\bar{\Delta}^j}(\mathcal{A}))(|z| - r_j^{\Delta^j}(\mathcal{A})) \leq r_i^{\Delta^j}(\mathcal{A})r_j^{\bar{\Delta}^j}(\mathcal{A}) \right\}.$$

Proof. Let λ be a Z -eigenvalue of \mathcal{A} with corresponding Z -eigenvector $x = (x_1, \dots, x_n)^T \in \mathbb{C}^n \setminus \{0\}$, i.e.,

$$\mathcal{A}x^{m-1} = \lambda x, \text{ and } \|x\|_2 = 1. \quad (1)$$

Let $|x_t| = \max_{i \in N} |x_i|$. Obviously, $0 < |x_t|^{m-1} \leq |x_t| \leq 1$. For $\forall j \in N, j \neq t$, from (1), we have

$$\lambda x_t = \sum_{j \in \{i_2, \dots, i_m\}} a_{ti_2 \dots i_m} x_{i_2} \cdots x_{i_m} + \sum_{j \notin \{i_2, \dots, i_m\}} a_{ti_2 \dots i_m} x_{i_2} \cdots x_{i_m}.$$

Taking modulus in the above equation and using the triangle inequality gives

$$\begin{aligned} |\lambda| |x_t| &\leq \sum_{j \in \{i_2, \dots, i_m\}} |a_{ti_2 \dots i_m}| |x_{i_2}| \cdots |x_{i_m}| + \sum_{j \notin \{i_2, \dots, i_m\}} |a_{ti_2 \dots i_m}| |x_{i_2}| \cdots |x_{i_m}| \\ &\leq \sum_{j \in \{i_2, \dots, i_m\}} |a_{ti_2 \dots i_m}| |x_j| + \sum_{j \notin \{i_2, \dots, i_m\}} |a_{ti_2 \dots i_m}| |x_t| \\ &= r_t^{\Delta^j}(\mathcal{A}) |x_j| + r_t^{\bar{\Delta}^j}(\mathcal{A}) |x_t|, \end{aligned}$$

i.e.,

$$(|\lambda| - r_t^{\bar{\Delta}^j}(\mathcal{A})) |x_t| \leq r_t^{\Delta^j}(\mathcal{A}) |x_j|. \quad (2)$$

If $|x_j| = 0$, then $|\lambda| - r_t^{\bar{\Delta}^j}(\mathcal{A}) \leq 0$ as $|x_t| > 0$. When $|z| - r_j^{\Delta^j}(\mathcal{A}) \geq 0$, we have

$$(|\lambda| - r_t^{\bar{\Delta}^j}(\mathcal{A}))(|\lambda| - r_j^{\Delta^j}(\mathcal{A})) \leq 0 \leq r_t^{\Delta^j}(\mathcal{A})r_j^{\bar{\Delta}^j}(\mathcal{A}),$$

which implies $\lambda \in \bigcap_{j \in N, j \neq t} \tilde{\Omega}_{t,j}(\mathcal{A}) \subseteq \Omega(\mathcal{A})$ from the arbitrariness of j . When $|z| - r_j^{\Delta^j}(\mathcal{A}) < 0$, from the arbitrariness of j , we have $\lambda \in \bigcap_{j \in N, j \neq t} \hat{\Omega}_{t,j}(\mathcal{A}) \subseteq \Omega(\mathcal{A})$.

Otherwise, $|x_j| > 0$. From (1), we can get

$$\begin{aligned} |\lambda||x_j| &\leq \sum_{j \in \{i_2, \dots, i_m\}} |a_{ji_2 \dots i_m}| |x_{i_2}| \cdots |x_{i_m}| + \sum_{j \notin \{i_2, \dots, i_m\}} |a_{ji_2 \dots i_m}| |x_{i_2}| \cdots |x_{i_m}| \\ &\leq \sum_{j \in \{i_2, \dots, i_m\}} |a_{ji_2 \dots i_m}| |x_j| + \sum_{j \notin \{i_2, \dots, i_m\}} |a_{ji_2 \dots i_m}| |x_t| \\ &= r_j^{\Delta^j}(\mathcal{A})|x_j| + r_j^{\bar{\Delta}^j}(\mathcal{A})|x_t|, \end{aligned}$$

i.e.,

$$(|\lambda| - r_j^{\Delta^j}(\mathcal{A}))|x_j| \leq r_j^{\bar{\Delta}^j}(\mathcal{A})|x_t|. \quad (3)$$

By (2), it is not difficult to see $\lambda \in \mathcal{K}_t(\mathcal{A})$. When $|\lambda| - r_t^{\bar{\Delta}^j}(\mathcal{A}) \geq 0$ or $|\lambda| - r_j^{\Delta^j}(\mathcal{A}) \geq 0$ holds, multiplying (2) with (3) and noting that $|x_t||x_j| > 0$, we have

$$(|\lambda| - r_t^{\bar{\Delta}^j}(\mathcal{A}))(|\lambda| - r_j^{\Delta^j}(\mathcal{A})) \leq r_t^{\Delta^j}(\mathcal{A})r_j^{\bar{\Delta}^j}(\mathcal{A}),$$

which implies $\lambda \in \bigcap_{j \in N, j \neq t} (\tilde{\Omega}_{t,j}(\mathcal{A}) \cap \mathcal{K}_t(\mathcal{A})) \subseteq \Omega(\mathcal{A})$ from the arbitrariness of j . And when $|\lambda| - r_t^{\bar{\Delta}^j}(\mathcal{A}) < 0$ and $|\lambda| - r_j^{\Delta^j}(\mathcal{A}) < 0$ hold, from the arbitrariness of j , we have $\lambda \in \bigcap_{j \in N, j \neq t} \hat{\Omega}_{t,j}(\mathcal{A}) \subseteq \Omega(\mathcal{A})$. Hence, the conclusion $\sigma(\mathcal{A}) \subseteq \Omega(\mathcal{A})$ follows immediately from what we have proved. \square

Next, a comparison theorem is given for Theorems 1-4.

Theorem 5. Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m,n]}$. Then

$$\Omega(\mathcal{A}) \subseteq \Psi(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}).$$

Proof. From Theorem 5 in [5], we have $\Psi(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A})$. Hence, here only $\Omega(\mathcal{A}) \subseteq \Psi(\mathcal{A})$ is proved. Let $z \in \Omega(\mathcal{A})$. Then $z \in \bigcup_{i \in N} \bigcap_{j \in N, j \neq i} \hat{\Omega}_{i,j}(\mathcal{A})$ or $z \in \bigcup_{i \in N} \bigcap_{j \in N, j \neq i} (\tilde{\Omega}_{i,j}(\mathcal{A}) \cap \mathcal{K}_i(\mathcal{A}))$.

We next divide the proof into two cases.

Case I: If $z \in \bigcup_{i \in N} \bigcap_{j \in N, j \neq i} \hat{\Omega}_{i,j}(\mathcal{A})$, then there is one index $i \in N$ such that $|z| < r_i^{\bar{\Delta}^j}(\mathcal{A})$ and $|z| < r_j^{\Delta^j}(\mathcal{A}), \forall j \in N, j \neq i$. Then, it is easy to see that

$$(|z| - r_i^{\bar{\Delta}^j}(\mathcal{A}))|z| \leq 0 \leq r_i^{\Delta^j}(\mathcal{A})R_j(\mathcal{A}), \quad \forall j \in N, j \neq i,$$

which implies that $z \in \bigcap_{j \in N, j \neq i} \Psi_{i,j}(\mathcal{A}) \subseteq \Psi(\mathcal{A})$. This implies $\Omega(\mathcal{A}) \subseteq \Psi(\mathcal{A})$.

Case II: If $z \in \bigcup_{i \in N} \bigcap_{j \in N, j \neq i} \left(\tilde{\Omega}_{i,j}(\mathcal{A}) \cap \mathcal{K}_i(\mathcal{A}) \right)$, then there is one index $i \in N$, for any $j \in N, j \neq i$, such that

$$|z| \leq R_i(\mathcal{A}), \quad (4)$$

and

$$(|z| - r_i^{\bar{\Delta}^j}(\mathcal{A}))(|z| - r_j^{\Delta^j}(\mathcal{A})) \leq r_i^{\Delta^j}(\mathcal{A})r_j^{\bar{\Delta}^j}(\mathcal{A}). \quad (5)$$

(i) If $r_i^{\Delta^j}(\mathcal{A})r_j^{\bar{\Delta}^j}(\mathcal{A}) = 0$, then $|z| \leq r_i^{\bar{\Delta}^j}(\mathcal{A})$ or $|z| \leq r_j^{\Delta^j}(\mathcal{A})$. When $|z| \leq r_i^{\bar{\Delta}^j}(\mathcal{A})$, we have

$$(|z| - r_i^{\bar{\Delta}^j}(\mathcal{A}))|z| \leq 0 \leq r_i^{\Delta^j}(\mathcal{A})R_j(\mathcal{A}),$$

which implies that $z \in \bigcap_{j \in N, j \neq i} \Psi_{i,j}(\mathcal{A}) \subseteq \Psi(\mathcal{A})$ from the arbitrariness of j . When $|z| \leq r_j^{\Delta^j}(\mathcal{A})$, we have

$$|z| \leq R_j(\mathcal{A}). \quad (6)$$

From (4), we can get

$$|z| - r_i^{\bar{\Delta}^j}(\mathcal{A}) \leq r_i^{\Delta^j}(\mathcal{A}). \quad (7)$$

Multiplying (6) and (7), we have

$$(|z| - r_i^{\bar{\Delta}^j}(\mathcal{A}))|z| \leq r_i^{\Delta^j}(\mathcal{A})R_j(\mathcal{A}), \quad (8)$$

which also implies that $z \in \bigcap_{j \in N, j \neq i} \Psi_{i,j}(\mathcal{A}) \subseteq \Psi(\mathcal{A})$, consequently, $\Omega(\mathcal{A}) \subseteq \Psi(\mathcal{A})$.

(ii) If $r_i^{\Delta^j}(\mathcal{A})r_j^{\bar{\Delta}^j}(\mathcal{A}) > 0$, then dividing both sides by $r_i^{\Delta^j}(\mathcal{A})r_j^{\bar{\Delta}^j}(\mathcal{A})$ in (5), we have

$$\frac{|z| - r_i^{\bar{\Delta}^j}(\mathcal{A})}{r_i^{\Delta^j}(\mathcal{A})} \frac{|z| - r_j^{\Delta^j}(\mathcal{A})}{r_j^{\bar{\Delta}^j}(\mathcal{A})} \leq 1. \quad (9)$$

From (4), we can get (7) and furthermore $\frac{|z| - r_i^{\bar{\Delta}^j}(\mathcal{A})}{r_i^{\Delta^j}(\mathcal{A})} \leq 1$. When $\frac{|z| - r_j^{\Delta^j}(\mathcal{A})}{r_j^{\bar{\Delta}^j}(\mathcal{A})} \leq 1$, then (6) holds. Multiplying (6) and (7), we can get (8), which implies that $z \in \bigcap_{j \in N, j \neq i} \Psi_{i,j}(\mathcal{A}) \subseteq \Psi(\mathcal{A})$, consequently, $\Omega(\mathcal{A}) \subseteq \Psi(\mathcal{A})$.

And when $\frac{|z| - r_i^{\bar{\Delta}^j}(\mathcal{A})}{r_i^{\Delta^j}(\mathcal{A})} > 1$, we can obtain $|z| > R_j(\mathcal{A})$. Let $a = |z|, b = r_j^{\Delta^j}(\mathcal{A}) - |a_{jj\dots j}|, c = |a_{jj\dots j}|$ and $d = r_j^{\bar{\Delta}^j}(\mathcal{A})$. By Lemma 2.3 in [12], we have

$$\frac{|z|}{R_j(\mathcal{A})} = \frac{a}{b+c+d} \leq \frac{a-(b+c)}{d} = \frac{|z| - r_j^{\Delta^j}(\mathcal{A})}{r_j^{\bar{\Delta}^j}(\mathcal{A})}. \quad (10)$$

If $|z| > r_i^{\overline{\Delta}^j}(\mathcal{A})$, by (9) and (10), we have

$$\frac{|z| - r_i^{\overline{\Delta}^j}(\mathcal{A})}{r_i^{\Delta^j}(\mathcal{A})} \frac{|z|}{R_j(\mathcal{A})} \leq \frac{|z| - r_i^{\overline{\Delta}^j}(\mathcal{A})}{r_i^{\Delta^j}(\mathcal{A})} \frac{|z| - r_j^{\Delta^j}(\mathcal{A})}{r_j^{\overline{\Delta}^j}(\mathcal{A})} \leq 1,$$

equivalently,

$$(|z| - r_i^{\overline{\Delta}^j}(\mathcal{A}))|z| \leq r_i^{\Delta^j}(\mathcal{A})R_j(\mathcal{A}),$$

which implies that $z \in \bigcap_{j \in N, j \neq i} \Psi_{i,j}(\mathcal{A}) \subseteq \Psi(\mathcal{A})$ from the arbitrariness of j . If $|z| \leq r_i^{\overline{\Delta}^j}(\mathcal{A})$, we have

$$(|z| - r_i^{\overline{\Delta}^j}(\mathcal{A}))|z| \leq 0 \leq r_i^{\Delta^j}(\mathcal{A})R_j(\mathcal{A}).$$

This also leads to $z \in \bigcap_{j \in N, j \neq i} \Psi_{i,j}(\mathcal{A}) \subseteq \Psi(\mathcal{A})$, consequently, $\Omega(\mathcal{A}) \subseteq \Psi(\mathcal{A})$. The conclusion follows from Case I and Case II. \square

Remark 1. Theorem 5 shows that the set $\Omega(\mathcal{A})$ in Theorem 4 is tighter than $\mathcal{K}(\mathcal{A})$ in Theorem 1, $\mathcal{L}(\mathcal{A})$ in Theorem 2 and $\Psi(\mathcal{A})$ in Theorem 3, that is, $\Omega(\mathcal{A})$ can capture all Z -eigenvalues of \mathcal{A} more precisely than $\mathcal{K}(\mathcal{A})$, $\mathcal{L}(\mathcal{A})$ and $\Psi(\mathcal{A})$.

Now, an example is given to verify the fact in Remark 2.

Example 1. Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,2]}$ be a symmetric tensor defined by

$$a_{1111} = 1, a_{1112} = 1, a_{1122} = 0.25, a_{2222} = 5, \text{ and } a_{ijkl} = 0 \text{ elsewhere.}$$

By computation, we get that all the Z -eigenvalues of \mathcal{A} are $-0.2044, -0.2044, 5.0000$ and 5.0000 . By Theorem 1, we have

$$\mathcal{K}(\mathcal{A}) = \{z \in \mathbb{C} : |z| \leq 6.7500\}.$$

By Theorem 2, we have

$$\mathcal{L}(\mathcal{A}) = \{z \in \mathbb{C} : |z| \leq 6.4827\}.$$

By Theorem 3, we have

$$\Psi(\mathcal{A}) = \{z \in \mathbb{C} : |z| \leq 6.3161\}.$$

By Theorem 4, we have

$$\Omega(\mathcal{A}) = \{z \in \mathbb{C} : |z| \leq 5.0000\}.$$

The Z -eigenvalue inclusion sets $\mathcal{K}(\mathcal{A})$, $\mathcal{L}(\mathcal{A})$, $\Psi(\mathcal{A})$, $\Omega(\mathcal{A})$ and the exact Z -eigenvalues are drawn in Figure 1, where $\mathcal{K}(\mathcal{A})$, $\mathcal{L}(\mathcal{A})$, $\Psi(\mathcal{A})$ and $\Omega(\mathcal{A})$ are represented by black dashed boundary, green solid boundary, blue point line boundary and red solid boundary, respectively. The exact eigenvalues are plotted by black “+”. It is easy to see $\sigma(\mathcal{A}) \subseteq \Omega(\mathcal{A}) \subset \Psi(\mathcal{A}) \subset \mathcal{L}(\mathcal{A}) \subset \mathcal{K}(\mathcal{A})$, that is, $\Omega(\mathcal{A})$ can capture all Z -eigenvalues of \mathcal{A} more precisely than $\Psi(\mathcal{A})$, $\mathcal{L}(\mathcal{A})$ and $\mathcal{K}(\mathcal{A})$.

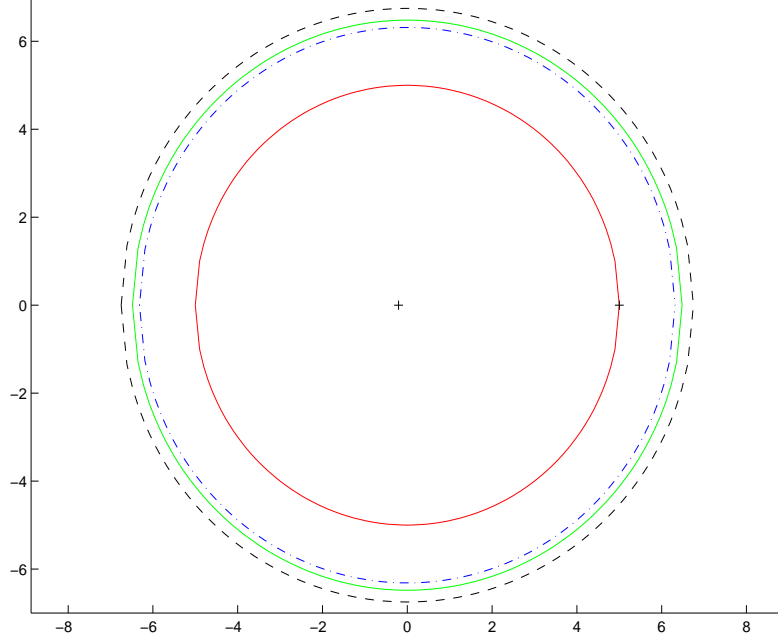


Figure 1: Comparisons of $\mathcal{K}(\mathcal{A})$, $\mathcal{L}(\mathcal{A})$, $\Psi(\mathcal{A})$ and $\Omega(\mathcal{A})$.

3 A sharper upper bound for the Z -spectral radius of weakly symmetric nonnegative tensors

As the Z -spectral radius of weakly symmetric nonnegative tensors plays a fundamental role in the symmetric best rank-one approximation [10, 16], recently, many people focus on bounding the Z -spectral radius of weakly symmetric nonnegative tensors. As an application of the set in Theorem 4, we in this section give a sharper upper bound for the Z -spectral radius of weakly symmetric nonnegative tensors.

Theorem 6. *Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m, n]}$ be a weakly symmetric nonnegative tensor. Then*

$$\varrho(\mathcal{A}) \leq \Omega_{\max}(\mathcal{A}) = \max \{ \hat{\Omega}_{\max}(\mathcal{A}), \tilde{\Omega}_{\max}(\mathcal{A}) \},$$

where

$$\begin{aligned} \hat{\Omega}_{\max}(\mathcal{A}) &= \max_{i \in N} \min_{j \in N, j \neq i} \min \{ r_i^{\bar{\Delta}_j}(\mathcal{A}), r_j^{\Delta_j}(\mathcal{A}) \}, \\ \tilde{\Omega}_{\max}(\mathcal{A}) &= \max_{i \in N} \min_{j \in N, j \neq i} \min \{ R_i(\mathcal{A}), \bar{\Omega}_{i,j}(\mathcal{A}) \}, \end{aligned}$$

and

$$\bar{\Omega}_{i,j}(\mathcal{A}) = \frac{1}{2} \left\{ r_i^{\bar{\Delta}_j}(\mathcal{A}) + r_j^{\Delta_j}(\mathcal{A}) + \sqrt{(r_i^{\bar{\Delta}_j}(\mathcal{A}) - r_j^{\Delta_j}(\mathcal{A}))^2 + 4r_i^{\Delta_j}(\mathcal{A})r_j^{\bar{\Delta}_j}(\mathcal{A})} \right\}.$$

Proof. By Lemma 4.4 in [4], we know that $\varrho(\mathcal{A})$ is a Z -eigenvalue of \mathcal{A} . By Theorem 4, we have

$$\varrho(\mathcal{A}) \in \bigcup_{i \in N} \bigcap_{j \in N, j \neq i} \hat{\Omega}_{i,j}(\mathcal{A}) \text{ or } \varrho(\mathcal{A}) \in \bigcup_{i \in N} \bigcap_{j \in N, j \neq i} \left(\tilde{\Omega}_{i,j}(\mathcal{A}) \cap \mathcal{K}_i(\mathcal{A}) \right).$$

If $\varrho(\mathcal{A}) \in \bigcup_{i \in N} \bigcap_{j \in N, j \neq i} \hat{\Omega}_{i,j}(\mathcal{A})$, then there is one index $i \in N$ such that

$$\varrho(\mathcal{A}) < r_i^{\bar{\Delta}^j}(\mathcal{A}) \text{ and } \varrho(\mathcal{A}) < r_j^{\Delta^j}(\mathcal{A}), \forall j \in N, j \neq i.$$

Then we have $\varrho(\mathcal{A}) \leq \min_{j \in N, j \neq i} \min \{r_i^{\bar{\Delta}^j}(\mathcal{A}), r_j^{\Delta^j}(\mathcal{A})\}$. Furthermore, we have

$$\varrho(\mathcal{A}) \leq \max_{i \in N} \min_{j \in N, j \neq i} \min \{r_i^{\bar{\Delta}^j}(\mathcal{A}), r_j^{\Delta^j}(\mathcal{A})\}.$$

If $\varrho(\mathcal{A}) \in \bigcup_{i \in N} \bigcap_{j \in N, j \neq i} \left(\tilde{\Omega}_{i,j}(\mathcal{A}) \cap \mathcal{K}_i(\mathcal{A}) \right)$, then there is one index $i \in N$, for any $j \in N, j \neq i$, such that

$$\varrho(\mathcal{A}) \leq R_i(\mathcal{A}) \tag{11}$$

and

$$(\varrho(\mathcal{A}) - r_i^{\bar{\Delta}^j}(\mathcal{A}))(\varrho(\mathcal{A}) - r_j^{\Delta^j}(\mathcal{A})) \leq r_i^{\Delta^j}(\mathcal{A})r_j^{\bar{\Delta}^j}(\mathcal{A}).$$

Solving $\varrho(\mathcal{A})$ in above inequality gives

$$\varrho(\mathcal{A}) \leq \frac{1}{2} \left\{ r_i^{\bar{\Delta}^j}(\mathcal{A}) + r_j^{\Delta^j}(\mathcal{A}) + \sqrt{(r_i^{\bar{\Delta}^j}(\mathcal{A}) - r_j^{\Delta^j}(\mathcal{A}))^2 + 4r_i^{\Delta^j}(\mathcal{A})r_j^{\bar{\Delta}^j}(\mathcal{A})} \right\} = \bar{\Omega}_{i,j}(\mathcal{A}). \tag{12}$$

Combining (11) and (12), and by the arbitrariness of j , we have

$$\varrho(\mathcal{A}) \leq \min_{j \in N, j \neq i} \min \{R_i(\mathcal{A}), \bar{\Omega}_{i,j}(\mathcal{A})\} \leq \max_{i \in N} \min_{j \in N, j \neq i} \min \{R_i(\mathcal{A}), \bar{\Omega}_{i,j}(\mathcal{A})\}.$$

The conclusion follows from what we have proved. \square

By Corollary 4.1 of [4], Theorem 6 of [5] and Theorem 5, the following comparison theorem can be derived easily.

Theorem 7. Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m,n]}$ be a weakly symmetric nonnegative tensor. Then the upper bound in Theorem 6 is smaller than those in Theorem 5 of [5], Theorem 4.5 of [4] and Corollary 4.5 of [6], that is,

$$\begin{aligned} \Omega_{\max}(\mathcal{A}) &\leq \max_{i \in N} \min_{j \in N, j \neq i} \frac{1}{2} \left\{ r_i^{\bar{\Delta}^j}(\mathcal{A}) + \sqrt{(r_i^{\bar{\Delta}^j}(\mathcal{A}))^2 + 4r_i^{\Delta^j}(\mathcal{A})R_j(\mathcal{A})} \right\} \\ &\leq \max_{i \in N} \min_{j \in N, j \neq i} \frac{1}{2} \left\{ R_i(\mathcal{A}) - a_{ij \dots j} + \sqrt{(R_i(\mathcal{A}) - a_{ij \dots j})^2 + 4a_{ij \dots j}R_j(\mathcal{A})} \right\} \\ &\leq \max_{i \in N} R_i(\mathcal{A}). \end{aligned}$$

Finally, we show that the upper bound in Theorem 6 is smaller than those in [4–10] by the following example.

Example 2. Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{[3,3]}$ be a weakly symmetric nonnegative tensor with entries defined as follows:

$$\mathcal{A}(:, :, 1) = \begin{pmatrix} 3 & 3 & 0 \\ 3 & 2 & 2.5 \\ 0.5 & 2.5 & 0 \end{pmatrix}, \quad \mathcal{A}(:, :, 2) = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 0 & 3 \\ 2.5 & 3 & 1 \end{pmatrix}, \quad \mathcal{A}(:, :, 3) = \begin{pmatrix} 1 & 3 & 0 \\ 2.5 & 3 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

By Corollary 4.5 of [6] and Theorem 3.3 of [7], we both have

$$\varrho(\mathcal{A}) \leq 19.$$

By Theorem 3.5 of [8], we have

$$\varrho(\mathcal{A}) \leq 18.6788.$$

By Theorem 4.6 of [4], we have

$$\varrho(\mathcal{A}) \leq 18.6603.$$

By Theorem 4.5 of [4] and Theorem 6 of [9], we both have

$$\varrho(\mathcal{A}) \leq 18.5656.$$

By Theorem 4.7 of [4], we have

$$\varrho(\mathcal{A}) \leq 18.3417.$$

By Theorem 2.9 of [10], we have

$$\varrho(\mathcal{A}) \leq 17.2063.$$

By Theorem 5 of [5], we obtain

$$\varrho(\mathcal{A}) \leq 15.2580,$$

By Theorem 6, we obtain

$$\varrho(\mathcal{A}) \leq 14.9410.$$

This example shows that the bound in Theorem 6 is the smallest.

Remark 2. From Example 1, it is not difficult to see that the upper bound in Theorem 6 could reach the true value of $\varrho(\mathcal{A})$ in some cases.

4 Conclusion

In this paper, we present a new Z -eigenvalue localization set $\Omega(\mathcal{A})$ and prove that this set is tighter than those in [4, 5]. As an application, we obtain a new upper bound $\Omega_{max}(\mathcal{A})$ for the Z -spectral radius of weakly symmetric nonnegative tensors, and show that this bound is sharper than those in [4–10] in some cases by a numerical example.

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